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Interpolating ensembles of random unitary matrices

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We investigate transitions between ensembles of unitary random matrices modeling changes of statistical properties of quantum chaotic systems which are periodically time dependent. The transitions simulating change of integrability are modeled with the help of numerically generated one-parameter families of ensembles interpolating between an ensemble of diagonal unitary matrices and a circular unitary or circular orthogonal ensemble. In an analogous manner we describe transitions between circular orthogonal and unitary ensembles corresponding to the time-reversal symmetry-breaking perturbation of a chaotic system. In all cases we present results concerning statistics of the quasienergy levels and eigenvectors.

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I. INTRODUCTION

In the previous paper [1] we presented a simple method of generating matrices representative for circular random matrix ensembles of Dyson, i.e., circular unitary ensemble (CUE) and circular orthogonal ensemble (COE) [2,3]. Since matrix elements of unitary matrices are correlated, construction of random unitary matrices is more complicated than construction of Hermitian matrices typical to Gaussian ensembles. Statistical properties and correlations between matrix elements of COE and CUE matrices where analyzed by Pereyra and Mello [4].

Numerous investigations showed that various spectral properties of propagators of time periodic, quantum chaotic systems [5], or scattering systems [6,7] are well described in terms of matrices belonging to these statistical ensembles if the corresponding classical systems are fully chaotic, i.e., the dominant part of the phase space is covered by chaotic trajectories. A typical classical dynamical system is neither fully chaotic nor integrable [8]. It is thus guessed that statistical properties of the spectra of such systems should interpolate between those for integrable and chaotic systems. The relevant hypothesis was proposed by Berry and Robnik in the case of autonomous systems [9]. In their approach, the spectrum mixes properties of integrable and chaotic systems in proportion dependent of the relative volumes of the classical phase space covered by regular and chaotic motions. In the case of a system that depends parametrically on the perturbation parameter, it is interesting to connect the properties of the spectra to the strength of perturbation, which causes nonintegrability [10].

It is easy to imagine yet another situation in which statistical properties of the spectrum do not conform to the description in terms of one of the above mentioned ensembles. Let us imagine that the time-reversal symmetry of the system in consideration is weakly broken by some perturbation controlled by the parameter λ [11]. Changing the magnitude of the parameter, we observe a gradual change of the degree of repulsion between neighboring levels (as well as other statistical properties of the spectrum). As in the case of a transition from an integrable to a chaotic system, the dependence of the properties of the spectrum on the strength of the perturbation is of main interest here.

Numerical investigations of the various transitions between universality classes of level repulsion were performed for different genuine dynamical systems. The investigations concentrated predominantly on time dependent, periodic systems (kicked rotator [12]), for which the description in terms of the circular ensembles of unitary matrices is appropriate. In an analogous study performed for the model of periodically kicked top [13], it was shown that the transition can differ in some respects from the one taking place in the similar situation but in the Gaussian ensembles corresponding to autonomous systems.

The universality of results for autonomous systems can be checked by investigation of analogous transitions using numerically generated random Hermitian matrices *in lieu* of a genuine Hamiltonian. In the present paper, we propose the similar procedure for investigating transitions in the circular ensembles by generating appropriate random matrices. In contrast to the earlier discussed ensembles interpolating between circular ensembles of random matrices [14,15], our scheme provides an explicit algorithm allowing one to study numerically the transitions between ensembles of unitary random matrices.

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II. INTERPOLATING ENSEMBLES FOR TRANSITIONS BETWEEN INTEGRABLE AND NONINTEGRABLE SYSTEMS

A. Dyson circular ensembles

In order to investigate properties of transitions between integrable and nonintegrable systems, we must give a random matrix model of an integrable time-periodic system. It is known that spectral properties of eigenphases of such a system are well described by a set of random unimodular numbers, phases of which are uniformly distributed on the circle. It suggests that an appropriate ensemble should consist of $N \times N$ diagonal matrices with diagonal elements equal to $\exp(i\phi_i)$, where $\phi_i, i = 1, \ldots, N$, are independent real random variables uniformly distributed from 0 to 2π. Let us define

$$\begin{aligned} &D = \mathrm{diag}(a_1, a_2, \ldots, a_N), \\ &a_i = e^{i\phi_i}, \quad 0 \le \phi_i < 2\pi. \end{aligned} \tag{2.1}$$

Since the randomly taken eigenphases are generated according to the Poisson process, we call the above ensemble circular poisson (CPE). For large N, the level spacing statistics $P(s)$ conform to the exponential distribution, typical to quantum analogues of generic classically regular systems.

The construction of matrices representing circular unitary ensemble based on the Hurwitz parametrization [16] was presented in the previous paper. Here we recapitulate the procedure briefly. An arbitrary unitary transformation U can be composed from elementary unitary transformations in two-dimensional subspaces. The matrix of such an elementary unitary transformation will be denoted by $E^{(i,j)}(\phi,\psi,\chi)$. The only nonzero elements of $E^{(i,j)}$ are

$$\begin{aligned} E_{kk}^{(i,j)} &= 1, \quad k = 1, \ldots, N; \quad k \neq i, j, \\ E_{ii}^{(i,j)} &= \cos(\phi) e^{i\psi}, \\ E_{ij}^{(i,j)} &= \sin(\phi) e^{i\chi}, \\ E_{ji}^{(i,j)} &= -\sin(\phi) e^{-i\chi}, \\ E_{jj}^{(i,j)} &= \cos(\phi) e^{-i\psi}. \end{aligned} \tag{2.2}$$

From the above elementary unitary transformations, one constructs the following $N-1$ composite rotations:

$$\begin{aligned} E_1 &= E^{(1,2)}(\phi_{12}, \psi_{12}, \chi_{12}), \\ E_2 &= E^{(2,3)}(\phi_{23}, \psi_{23}, 0) E^{(1,3)}(\phi_{13}, \psi_{13}, \chi_{13}), \\ E_3 &= E^{(3,4)}(\phi_{34}, \psi_{34}, 0) E^{(2,4)}(\phi_{24}, \psi_{24}, 0) \\ &\quad \times E^{(1,4)}(\phi_{14}, \psi_{14}, \chi_{14}), \\ &\cdots \\ E_{N-1} &= E^{(N-1,N)}(\phi_{N-1,N}, \psi_{N-1,N}, 0) \\ &\quad \times E^{(N-2,N)}(\phi_{N-2,N}, \psi_{N-2,N}, 0) \\ &\quad \times \cdots E^{(1,N)}(\phi_{1N}, \psi_{1N}, \chi_{1N}), \end{aligned} \tag{2.3}$$

and finally forms the unitary transformation U as

$$U = e^{i\alpha} E_1 E_2 E_3 \cdots E_{N-1}. \tag{2.4}$$

If the angles $\alpha, \phi_{rs}, \psi_{rs}$, and χ_{1s} are taken uniformly from the intervals

$$0 \le \psi_{rs} < 2\pi, \quad 0 \le \chi_{1s} < 2\pi, \quad 0 \le \alpha < 2\pi, \tag{2.5}$$

whereas

$$\phi_{rs} = \arcsin(\xi_{rs}^{1/2r}), \quad r = 1, 2, \ldots, N-1, \tag{2.6}$$

with ξ_{rs} uniformly distributed in

$$0 \le \xi_{rs} < 1, \tag{2.7}$$

then the obtained matrix is drawn from the CUE [1].

B. Poisson-unitary transitions

In order to imitate a one-parameter transition between the integrable and nonintegrable case, i.e., between diagonal matrices with uniformly and independently distributed elements [cf. Eq. (2.1)] and matrices from CUE, we propose the following construction. Instead of allowing the parameters ψ_{rs}, χ_{1s}, α, and ξ_{rs} to take their values from the intervals given by formulas (2.5) and (2.7), we restrict them to

$$\begin{aligned} &0 \le \psi_{rs} < 2\pi\delta, \quad 0 \le \chi_{1s} < 2\pi\delta, \\ &0 \le \alpha < 2\pi\delta, \quad 0 \le \xi_{rs} < \delta, \end{aligned} \tag{2.8}$$

keeping the rule (2.6) for generating ϕ_{rs}. The parameter δ controlling the transition takes values between zero and one. The constructed matrix, denoted in the following by $U_0(\delta)$, reduces to the unit matrix for $\delta = 0$ and becomes a CUE matrix for $\delta = 1$, when formula (2.8) reduces to (2.5) and (2.7). It is thus obvious that the one-parameter family of matrices

$$U(\delta) = D U_0(\delta), \tag{2.9}$$

interpolates between diagonal and circular unitary ensembles, since an additional multiplication of a CUE matrix $U_0(1)$ by a random diagonal one D does not change its statistical properties.

In order to achieve reliable statistics, we have constructed numerically 500 matrices of the size $N = 100$ with various values of the transition parameter δ, diagonalized them, and analyzed the properties of obtained eigenvalues and eigenvectors. To study the long range correlations of the spectrum, we computed the average number of levels $\langle N_S(L) \rangle$ in an interval of the length L, and the number variance $\Sigma^2(L) = \langle N_S^2(L) \rangle - \langle N_S(L) \rangle^2$. For an uncorrelated spectrum of Poisson ensemble, the number variance grows linearly with L, whereas correlation characteristic to unitary ensemble manifest itself in logarithmic growth of Σ^2 [3,17]

$$\Sigma_U^2(L) = \frac{1}{\pi^2} (\ln(2\pi L) + 1 + \gamma), \tag{2.10}$$

where $\gamma \approx 0.577\ldots$ is the Euler constant.

Figure 1 shows the dependence of the number variance $\Sigma^2(L)$ for $\delta = 0.0$ (CPE), $0.05, 0.1, 0.3, 0.5, 0.7, 0.9$, and 1.0 (CUE). In spite of the fact that several interpolating formulas for number variance are known in the literature [14,18], none of them seems to be applicable to the ensemble discussed. Note the smooth change of the spectral properties from Poisson to CUE with the increase of the control parameter δ. It is worthwhile to stress that for no values of δ does the number variance coincide with the COE behavior [3]

$$\Sigma^2_O(L) = \frac{2}{\pi^2}\left(\ln(2\pi L) + 1 + \gamma - \frac{\pi^2}{8}\right), \qquad (2.11)$$

represented by the dotted line. In other words, the ensemble interpolating between Poisson and unitary ensembles does not visit the orthogonal ensemble, and the CPE-COE transition has to be treated separately.

Figure 2 presents level spacing distribution $P(s)$ for three chosen intermediate values of δ. The dashed line stands for the Poisson exponential distribution, which describes the data for $\delta = 0$. The solid line represents the Wigner surmise for unitary ensemble

$$P_U(s) = \frac{32}{\pi^2}s^2 \exp\left[-\frac{4s^2}{\pi}\right], \qquad (2.12)$$

which gives a good approximation to the CUE distribution for large N [3]. As seen in Fig. 2, the transition CPE-CUE is almost completed for $\delta = 0.9$. Level spacing distribution for this transition can be approximated by an empirical formula proposed by Izrailev [19].

Complementary information concerning the transition between canonical ensembles can be obtained by studying the properties of eigenvectors. It is known [20–22] that in the limit of large N, the statistics of eigenvectors components $y = |c_{lk}|^2$ for canonical ensembles is given by the χ^2_ν distribution

$$P_\nu(y) = \frac{(\nu/2)^{(\nu/2)}}{\Gamma(\nu/2)\langle y\rangle}\left(\frac{y}{\langle y\rangle}\right)^{\nu/2-1} \exp\left[-\frac{\nu y}{2\langle y\rangle}\right], \qquad (2.13)$$

where the number of degrees of freedom ν equals one for the orthogonal and two for the unitary ensemble. Since

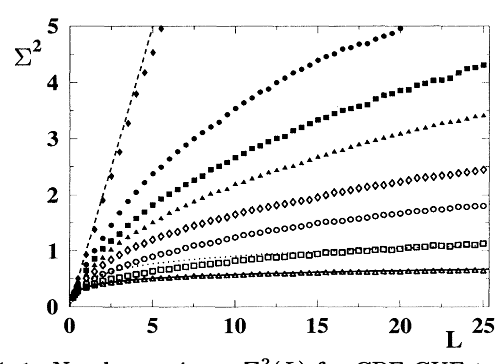

FIG. 1. Number variance $\Sigma^2(L)$ for CPE-CUE transition: $\delta = 0.0$ (♦), 0.05 (•), 0.1 (■), 0.3 (▲), 0.5 (◇), 0.7 (○), 0.9 (□), and 1.0 (△). Dashed, dotted, and solid lines represent behavior of Poisson, orthogonal, and unitary ensembles, respectively.

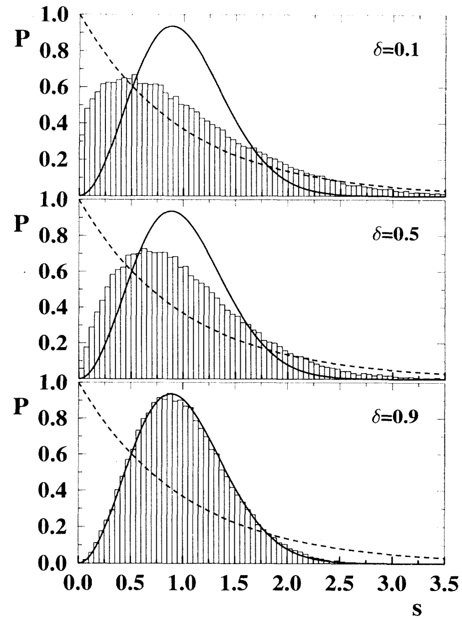

FIG. 2. Nearest neighbors distribution $P(s)$ for CPE (dashed line)-CUE (solid line) transition. Values of the parameter δ label each graph.

the distribution is peaked around zero, it is convenient to use a logarithmic scale and study $P(\log(y))$. Figure 3 shows the eigenvector distribution with the mean value $\langle y\rangle$ normalized to unity for $N = 100$ and the same three values of the parameter δ visualizing the transition CPE-CUE. For a small value of the parameter $\delta > 0$, the eigenvector statistics can be approximated by an ap-

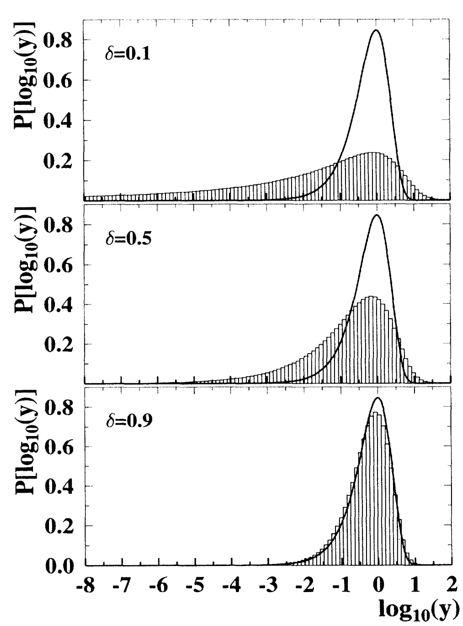

FIG. 3. Eigenvector statistics $P(\log_{10}(y))$ for the CPE-CUE transition for three intermediate values of the parameter δ. Solid line stands for $\chi^2_{\nu=2}$ distribution.

propriately normalized distribution $P_0(y) = 1/y$, which corresponds to a broad and flat distribution in the logarithmic scale. For larger values of δ, the width of the distribution $P(\log(y))$ gradually shrinks and the eigenvector statistics tends to the expected χ_2^2 distribution.

C. Poisson-orthogonal transitions

A similar construction can be done to model a transition between Poisson and circular orthogonal ensembles. To this end, let us remind our method of generating random unitary symmetric matrices representative to the circular orthogonal ensemble. In [1], we showed that it is enough to construct a random unitary matrix according to the prescription contained in the formulas (2.2)–(2.7) and take

$$S = U^T U, \tag{2.14}$$

with U^T denoting the transpose of U.

The generalization of the above construction consists thus of taking the previously described unitary matrix $U(\delta)$ given in (2.9) and defining

$$S(\delta) = [U(\delta)]^T U(\delta) \tag{2.15}$$

as a one-parameter family interpolating between diagonal $(D^2, \delta = 0)$ Poisson and circular orthogonal $[S = S(1), \delta = 1]$ ensembles.

Numerical construction of a COE unitary matrix is only slightly more time consuming than a CUE matrix of the same size. As in the previous case, our statistics consist of data obtained from 500 matrices of the size $N = 100$ with various values of the parameter δ controlling the CPE-COE transition. Figure 4 shows the dependence of the number variance $\Sigma^2(L)$ for $\delta = 0.0$ (CPE), $0.05, 0.1, 0.3, 0.5, 0.7, 0.9$, and 1.0 (COE). The limiting case may be approximated (for $L > 1$) by (2.11) represented by the solid line. As in the previously considered case, the transition occurs smoothly with variations of δ.

In an analogous way, we show in Figs. 5 and 6 the distribution of levels and eigenvectors for the CPE-COE transition. Observe that the changes of properties of the spectrum parallel the changes of the properties of the eigenvectors. Eigenvector statistics for this transition can be approximated by the χ_ν^2 distribution with a real parameter ν varying from zero to one, or by a more sophisticated distribution introduced for band symmetric matrices in Ref. [23].

Entire information on eigenvectors is included in the distribution $P(y)$. However, in order to describe the lo-

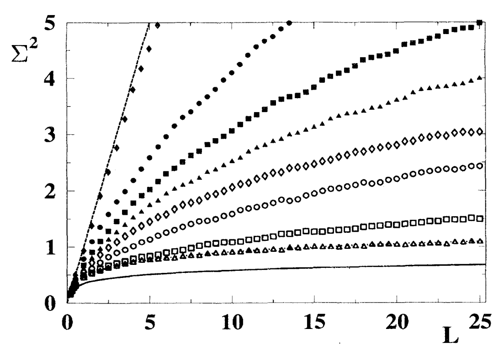

FIG. 4. As in Fig. 1 for the transition Poisson orthogonal.

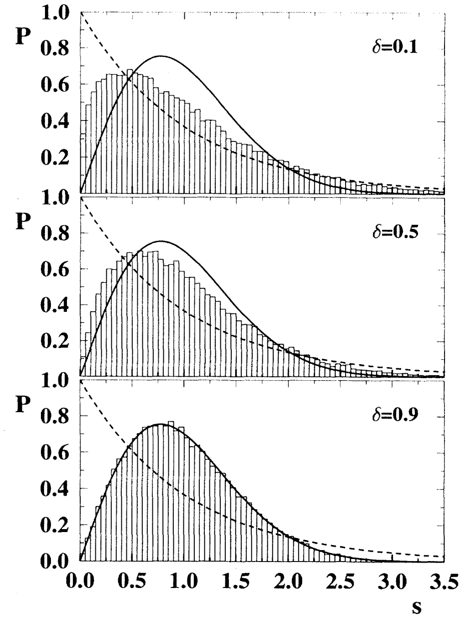

FIG. 5. Nearest neighbors distribution $P(s)$ for CPE (dashed line)-COE (solid line) transition. Values of the parameter δ label each graph.

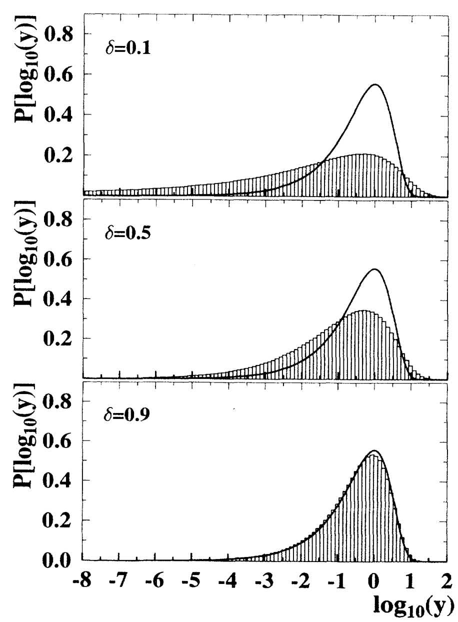

FIG. 6. Eigenvector statistics $P(\log_{10}(y))$ for the CPE-COE transition for three intermediate values of the parameter δ. Solid line stands for Porter-Thomas $(\chi_{\nu=1}^2)$ distribution.

calization properties of eigenvectors by a single quantity, it is convenient to define the Shannon entropy H_l of the eigenvector $|\varphi_l\rangle$

$$H_l := -\sum_{k=1}^{N} y_{lk} \ln(y_{lk}), \tag{2.16}$$

and to use the ensemble average $\langle H_S\rangle := \sum_{l=1}^{N} H_l/N$. Mean entropy of eigenvectors averaged over Dyson ensembles of unitary matrices can be found analytically [24] and expressed by means of the digamma function $\Psi(x)$ [25]

$$\langle H_\nu\rangle = \Psi\left(\frac{\nu N}{2}+1\right) - \Psi\left(\frac{\nu}{2}+1\right), \tag{2.17}$$

with $\nu = 1$ and 2 for COE and CUE, respectively.

When studying the transitions between canonical ensembles, one is interested in how the transition speed changes with the dimension of the matrices N. Defining the scaled entropy localization length as [26]

$$\xi_1 = \exp(\langle H_S\rangle - \langle H_1\rangle), \tag{2.18}$$

which varies from zero to unity, we may compare the properties of eigenvectors for the ensembles of different sizes. Figure 7 shows the dependence of the scaled localization length ξ_1 on the matrix size N for five values of the parameter δ controlling the CPE-COE transition. For each value of δ, the numerical data are distributed along horizontal lines, roughly at $\beta \approx \delta$. This provides a direct argument that transition speed depends only weakly on N. In order to investigate a transition in the properties of eigenvalues, we computed the number variance $\Sigma^2(1)$ for several values of transition parameter δ and observed its weak dependence on the size of matrices N. A similar result showing a weak dependence of the transition speed were also obtained for the Poisson-unitary transition.

III. TRANSITIONS BETWEEN DIFFERENT SYMMETRY CLASSES

As already mentioned, of special interest are situations in which a nonintegrable system, which is originally time-reversal invariant, is influenced by a perturbation, which is itself time-reversal breaking. For time-periodic systems, we encounter thus a transition between circular orthogonal and circular unitary ensemble. Generalizing our previous constructions of appropriate unitary and unitary symmetric matrices by introducing the one-parameter interpolating families, we consider

$$V(\delta) = U_0^T(1-\delta)U_0(1), \tag{3.1}$$

with $U_0(1-\delta)$ the previously described unitary matrix. Changing δ from zero when $V = U_0^T(1)U_0(1) = V^T$ to one when V becomes simply $U_0(1)$ we simulate a transition between COE and CUE.

In contrast to the other discussed transitions (CPE-CUE and CPE-COE), the transition does not occur in a uniform way with respect to the changes of the control parameter δ. The critical value of the parameter δ sufficient to complete the transition to CUE decreases with the matrix size.

In order to look for a scaling parameter $\lambda = \lambda(N,\delta)$, which governs the transition, we studied the log variance $\sigma^2_{\ln}$ of the eigenvector distribution $P(y)$. This quantity $\sigma^2_{\ln} = \langle \ln^2(y)\rangle - \langle \ln(y)\rangle^2$ is equal to $\pi^2/2 \approx 4.935$ for COE and decreases to $\pi^2/6 \approx 1.645$ for CUE [13] and can be used as a signature of the transition. We define the critical value the control parameter δ_c by a condition that the log variance is equal to an arbitrarily set intermediate value of 2.0

Figure 8 presents the dependence of the critical parameter δ_c on the matrix size N represented in the log-log scale. All numerical data are localized close to a line with the slope -1.48. This observation suggests that we assume an existence of scaling and introduce a scaled parameter $\lambda := \delta N^{3/2}$.

To verify the above scaling hypothesis, we computed eigenvector statistics for several values of λ and four different sizes of matrices. The distribution $P(y)$ has been characterized by the log variance $\sigma_L^2 n$ and the entropy localization length ξ_2 scaled with respect to CUE as $\xi_2 = \exp(\langle H_S\rangle - \langle H_2\rangle)$, which varies from $2/e$ (COE) to unity (CUE), according to Eq. (2.17).

The dependence of both quantities on λ is shown in Fig. 9. The data obtained for N varying from 30 to 240

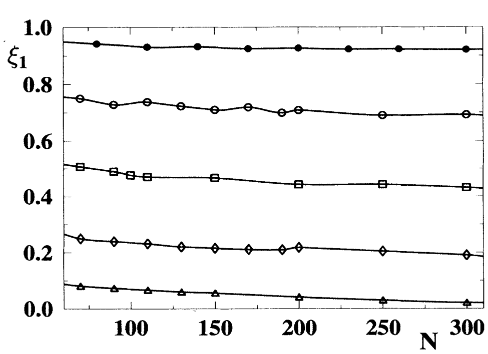

FIG. 7. Dependence of the entropy localization length ξ_1 of the eigenvectors of $N\times N$ matrices for the CPE-COE transition with $\delta = 0.1$ (△), 0.3 (◇), 0.5 (□), 0.7 (○), and 0.9 (•).

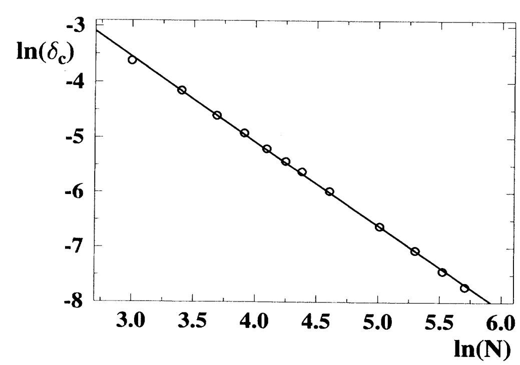

FIG. 8. Critical perturbation strength δ_c vs matrix size N in a log-log scale for COE-CUE transition. Linear fit gives the slope -1.48.

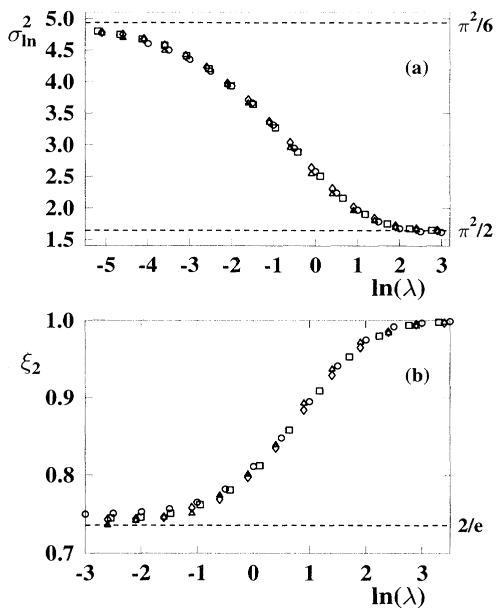

FIG. 9. Scaling of properties of eigenvectors for COE-CUE transition. (a) Log variance $\sigma^2_{\ln}$, and (b) localization length ξ_2 plotted as functions of scaling parameter $\lambda = \delta N^{3/2}$ for $N = 30$ (○), 60 (◇), 120 (□), and 240 (△). Horizontal lines represent the values typical of COE ($\pi^2/2$) and of CUE ($\pi^2/6$).

form a single line in each case, which confirms the conjecture that the transition speed of eigenvectors changes as $N^{-3/2}$. The scaled parameter λ varies per definition from zero to $N^{3/2}$, but the transition may be considered as completed for $\lambda \approx 20$. Log variance $\sigma^2_{\ln}$ is sensitive to small changes of λ close to zero while ξ_2 deviates from the COE value for $\lambda \sim 0.3$.

In order to give an independent argument supporting the found $N^{-3/2}$ scaling, we would like to present the following reasoning. The time-reversal symmetry is effectively broken when, due to changes of the parameter δ, the average shift of the eigenvalues is comparable to the mean spacing $\Delta s = 2\pi/N$ [5]. The average shift $\Delta\phi$ can be estimated as

$$\Delta\phi = \delta\sigma, \tag{3.2}$$

where σ^2 is the the mean-square velocity

$$\sigma = \sqrt{\left\langle \left(\frac{d\phi}{d\delta}\right)^2 \bigg|_{\delta=0} \right\rangle}. \tag{3.3}$$

The characteristic value of the perturbation parameter δ_c at which the transition effectively takes place can be thus written as

$$\delta_c = \frac{2\pi}{N}\frac{1}{\sigma}. \tag{3.4}$$

We calculated numerically the dependence of the mean-square velocity σ^2 on the size of the matrix N. The results presented in Fig. 10 show clearly that $\sigma^2 \sim N$. Therefore, we obtain

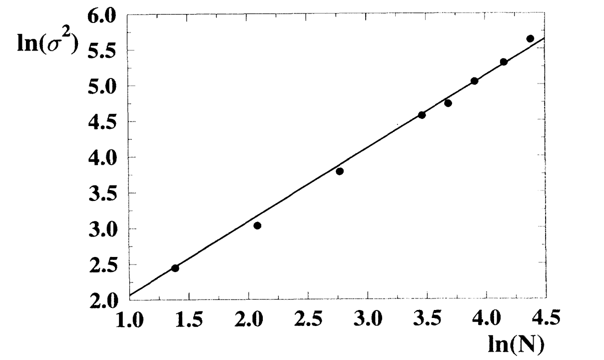

FIG. 10. The mean-squared velocity of levels $\sigma^2 := \langle (d\phi/d\delta)^2|_{\delta=0}\rangle$ vs the matrix dimension N for the COE-CUE transition. The fitted straightline corresponds to $\ln(\sigma^2) = 1.05 + 1.02\ln(N)$.

$$\delta_c \sim N^{-3/2}. \tag{3.5}$$

The presented model of the COE-CUE transition is obviously only a single representative of an infinite number of possible ones. In our investigations we were motivated by its simplicity.

The scaling parameter λ governs also the statistical properties of the spectrum. To show that, we plotted in Fig. 11 the number variance $\Sigma^2(L)$ obtained for $N = 30$ (samples of 2000 matrices) and $N = 50$ (samples of 1000 matrices) for four values of the scaled transition parameter λ. We have checked that for any value of λ the number variance $\Sigma^2(1)$ does not change much with N varying from 30 to 250. For $\lambda = 10.0$, the spectral properties of the ensemble are close to the behavior characteristic of CUE.

It might also be instructive to compare directly the changes of the properties of eigenvalues and eigenvectors during the transition COE-CUE. As shown in Figs. 12 and 13 (note the change of the horizontal scale with respect to Figs. 3 and 6) representing the level spacing distribution $P(s)$ and the eigenvector statistics $P(y)$ for

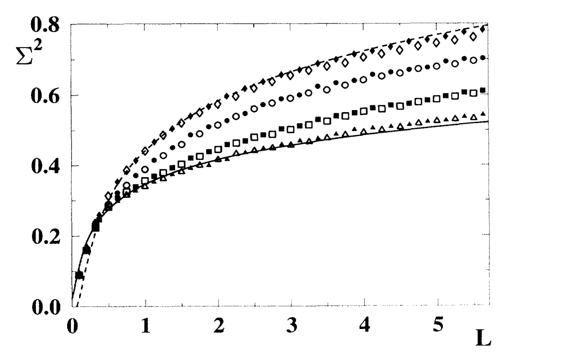

FIG. 11. Number variance $\Sigma^2(L)$ for COE-CUE transition: for $N = 30$ (open, larger symbols) and $N = 50$ (full, smaller symbols), and $\lambda = 0.3$ (◇), 2.0 (○), 5.0 (□), and 10.0 (△). Dotted and solid lines represent the behavior of orthogonal and unitary ensembles, respectively.

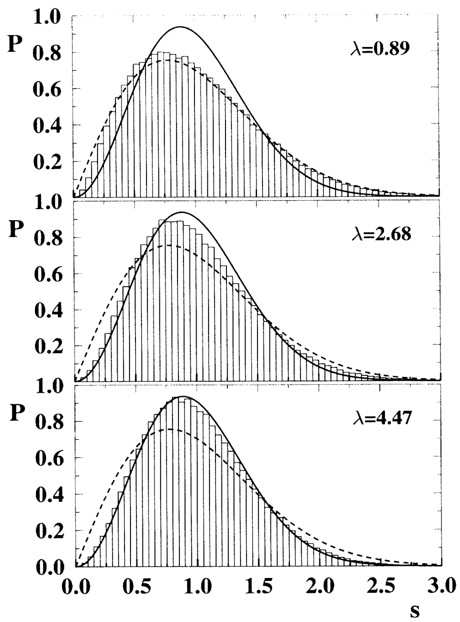

FIG. 12. Nearest neighbors distribution $P(s)$ for COE (dashed line)-CUE (solid line) transition for 2000 matrices of size $N = 20$. Values of the scaling parameter λ label each graph.

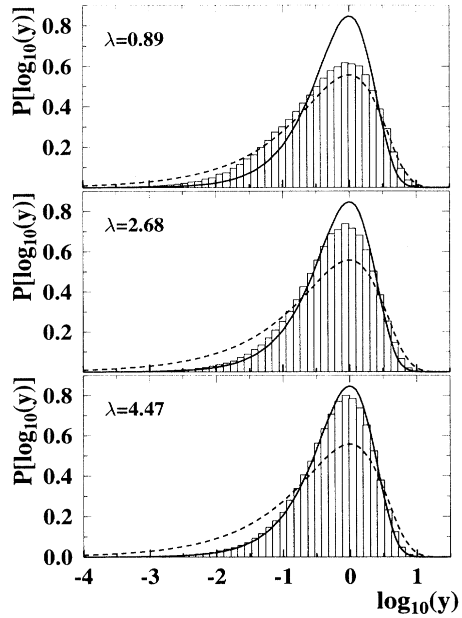

FIG. 13. Eigenvector statistics $P(\log(y))$ for the COE-CUE transition obtained for $N = 20$ matrices with three different values of the parameter λ. Solid line stands for $\chi^2_{\nu=2}$ and dashed line for $\chi^2_{\nu=1}$ distributions.

the same values of λ, the transition occurs in a parallel way. Level spacing distribution $P(s)$ for COE-CUE transition can be approximated by the Lenz-Haake distribution derived for 2×2 Hermitian matrices [10]. The simplest interpolating formula for eigenvector statistics $P(y)$ is given by χ^2_ν distribution with real parameter ν varying from one to two. Better approximation was proposed in [27], while the exact formula for $P(y)$ during the corresponding transition in the space of Hermitian matrices was found by Sommers and Iida [28]

IV. CONCLUDING REMARKS

Practical algorithms allowing one to construct numerically unitary matrices typical to circular unitary and orthogonal ensembles were presented in our previous paper [1]. In this work, we extended our model by introducing three one-parameter generalizations describing the transitions CPE-CUE, CPE-COE, and COE-CUE.

Transitions from regular to chaotic motion for time-periodic dynamical systems can be described by transitions from Poisson to orthogonal or unitary circular ensemble, depending on the symmetry of the system. Introduced interpolating ensembles of random unitary matrices change their properties in a smooth way with the control parameter. Moreover, in both cases the transition speed depends only weakly on the size of the matrix. These pleasant features of the defined ensembles allow us to hope that they might be successfully applied to mimic properties of periodic time-dependent dynamical systems or open scattering systems, described by an S matrix, in place of often used Gaussian ensembles of Hermitian matrices.

Time-reversal symmetry breaking in periodic dynamical systems corresponds to the transition between orthogonal to unitary circular ensembles. We have defined one of the possible ensembles realizing the transition COE-CUE and showed that the transition speed for *both* eigenvalues and eigenvectors scales as $N^{-3/2}$. This is in contrast to the analogous transition induced by a gradual breaking of the time-reversal symmetry in a dynamical model of periodically kicked top [13], for which properties of the spectrum change faster than properties of eigenvectors.

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